The Mis-use of the Vuong Test as a Test of Zero-Inflation

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The principal purpose of this talk is to alert the people in this room that "The Vuong Test for Non-Nested Models" is being widely mis-used as a test for zero-inflation.
Introduction

- The principal purpose of this talk is to alert the people in this room that "The Vuong Test for Non-Nested Models" is being widely mis-used as a test for zero-inflation,
- I will also outline how I believe this erroneous use has arisen,
- and introduce a diagramatic method for detecting zero-inflation.
Question:

Do you know what is meant by the phrase:

*Nested Models?*
Do You Agree?

The Gaussian Model:

\[ Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \]

is nested in the Gaussian Model:

\[ Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \]
Do You Agree?

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as the latter reduces to the former when \( \beta_3 = 0. \)
The Geometric Distribution:

\[ P(X = x) = (1 - p)^{x-1} p; \quad x = 1, 2, 3, \ldots \]

is nested in the Negative Binomial Distribution:

\[ P(X = x) = \binom{x - 1}{r - 1} (1 - p)^{x-r} p^r; \quad x = r, r + 1, r + 2, \ldots \]
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as the latter reduces to the former when \( r = 1 \).
Zero-Inflated Models?

Imagine I gave everyone here a marigold seed
What happens next?

- You go away and sow it
- In three months time you let me know how many flowers it has produced
- some will have produced one, two, three, four or more
- but some will have produced NONE
Two Types of None!!

- Your plant has grown, but there are no flowers
- The seed never germinated–you have no plant
- The first scenario is an example of a zero belonging to a count distribution, e.g. Poisson, distribution.
- The second is an example of a zero belonging to a “perfect zero” distribution.
Zero-Inflated Poisson Distribution

\[ P(X = x) = \begin{cases} 
\gamma + (1 - \gamma) \exp(-\theta) & X = 0 \\
(1 - \gamma) \frac{\exp(-\theta) \theta^x}{x!} & X = 1, 2, 3, \ldots
\end{cases} \]

In relation to our marigolds, it is the proportion of seeds that do not germinate, for example, if 23% of seeds fail to germinate, \( \gamma = 0.23 \). And similarly for other distributions.
The Vuong Test for Strictly Non-Nested Models

- In slightly simplified form, it states that under the null hypothesis that two non-nested models $F_\theta$ and $G_\gamma$ fit equally well, i.e. that the expected value of their log-likelihood ratio equals zero, then under $H_0$ the asymptotic distribution of the log-likelihood ratio statistic, $LR$, is normal.
The Vuong Test for Strictly Non-Nested Models

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- In particular, (under $H_0$):

$$
\frac{LR_n \left( \hat{\theta}_n, \hat{\gamma}_n \right)}{\hat{\omega}_n \sqrt{n}} \xrightarrow{n} N(0, 1) \quad (1)
$$

where $\omega$ denotes the variance of $LR_n$ and $n$ the sample size.
Vuong also presents a test for nested models, in which $-2 \times LR$ is compared to a central $\chi^2$ distribution, and a test for overlapping models where $-2 \times LR$ is compared to a weighted sum of $\chi^2$ distributions.
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(Due to the simplicity of its calculation, and the fact that it resembles a paired $t$-test?), the test has become popular among statistical practitioners in various disciplines and is implementable in Stata, and the R-package *pscl*.
The Vuong Test For Non-Nested Models is being widely used as a test for zero-inflation
The Vuong Test For Non-Nested Models is being widely used as a test for zero-inflation. Such use is advised by reputable sources.
R-package *pscl* help file says:

“The Vuong non-nested test is based on a comparison of the predicted probabilities of two models that do not nest. Examples include comparisons of zero-inflated count models with their non-zero-inflated analogs (e.g., zero-inflated Poisson versus ordinary Poisson, or zero-inflated negative-binomial versus ordinary negative-binomial).”
The Mis-use of the Vuong Test as a Test of Zero-Inflation

A website associated with UCLA

## Vuong Non-Nested Hypothesis Test-Statistic: -3.574
## (test-statistic is asymptotically distributed N(0,1)
## null that the models are indistinguishible)
## in this case:
## model2 > model1, with p-value 0.0001756

The Vuong test compares the zero-inflated model with an ordinary Poisson regression model. In this example, we can see that our test statistic is significant, indicating that the zero-inflated model is superior to the standard Poisson model.
I e-mailed them about this...
We agree with your first statement, that the Vuong statistic follows a standard normal distribution in the case of NON-NESTED models. However, we respectfully disagree with your assertion that the poisson model and the zero-inflated poisson models are nested. You cannot reduce a ZIP model down to a poisson model by setting the parameters in the zero-inflation model to 0. It is true that in order for a ZIP model and poisson model to be equivalent, the probability of being in the zero-generating group rather than the count-generating group must be zero.
However, because we are modeling logits/probits and not probabilities in the binary model, setting the parameters to 0 in the binary model will not set the probability of being in the zero-generating group to 0, but instead to 0.5 (standard normal \( \text{cdf}(0) = 0.5 \)). Thus, because the models are non-nested, we compare the Vuong statistic to the standard normal distribution.
Good Point But So What!:

\[
\exp(t) + \exp(-t) \neq 0 \quad \text{for all } t \in \mathbb{R}
\]

The models are "nested at infinity".
Good Point But So What!:

Whilst it is true that:

\[
\frac{\exp(t)}{1 + \exp(t)} \neq 0 \text{ for all } t \in \mathbb{R}
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and hence in some sense this formulation of the ZIP and Poisson are non-nested

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The models are “nested at infinity”
Good Point But So What!: 

• The models can be made as close to nested as you like.
• More technically this formulation of the zero-inflated model fails to meet Vuong's prerequisite that the parameter space is a compact subset of $\mathbb{R}^p$. 

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Why Is this Important

As a consequence of this:

- the sampling distribution of the zero-inflation parameter is non-normal
- This results on distribution of the log-likelihood ratios being non-normal
- Similar statements hold if probit or complementary log-log links are used.
In a Nutshell

There is confusion in the literature about models being nested if one reduces to the other if certain parameters are fixed, many authors apparently taking this to mean fixed at zero. For example, Desmarais and Harden state:

\textit{the count regression \( f \) is not “nested” in the zero-inflated model, because the model does not reduce to \( f \) when \( \gamma = 0 \), in which case the probability of a 0 is inflated by 0.50}

apparently alluding to the fact that the value of the logistic function \( = 0.5 \) when \( t = 0 \).
In a Nutshell

There is confusion in the literature about models being nested if one reduces to the other if certain parameters are fixed, many authors apparently taking this to mean fixed at zero. For example, Desmarais and Harden state:

*the count regression $f$ is not “nested” in the zero-inflated model, because the model does not reduce to $f$ when $\gamma = 0$, in which case the probability of a 0 is inflated by 0.50*

apparently alluding to the fact that the value of the logistic function $= 0.5$ when $t = 0$. (This would imply that the geometric distribution is not nested in the negative binomial)
Distributions of Vuong Statistic of Zero-Inflated Poisson versus Poisson Models

Simulated Distribution of the Vuong Statistic
H0: Poisson vs. H1 ZIP

Data Simulated from a Poisson(2) Distribution, no covariates
Value of Vuong Statistic, min value = −0.0025

Simulated Distribution of the Vuong Statistic
H0: Poisson vs. H1 ZIP

Data Simulated from a Poisson Distribution, One Covariate
Value of Vuong Statistic, min value = −0.00082
Distribution of Vuong Statistic Strictly Non-Nested Models

Using exactly the same code, but with strictly non-nested models:
Distribution of Vuong Statistic Strictly Non-Nested Models

Using exactly the same code, but with strictly non-nested models:

**Figure**: Distributions of the Log-likelihood Ratios of Strictly Non-Nested Models

Simulated Distribution of the Vuong Statistic
Strictly Non-Nested Poisson Models

Density
If we temporarily ignore the issue of whether zero-inflated models and their non-zero-inflated counterparts are non-nested or otherwise, and consider them non-nested, to appropriately simulate the distribution of the log-likelihood ratios it would be necessary to resample from data that was somehow equidistant from zero-inflated and non-zero-inflated data, it is difficult to envisage the nature of such data.
Non-rejection of the null hypothesis of Vuong’s test for non-nested models, where the (supposedly) non-nested models are say the zero-inflated Poisson and standard Poisson model would mean that there is no evidence to conclude that either model fits the data better than the other, not that there is no evidence to support zero-inflation, and its rejection simply implies that either the zero-inflated Poisson model fits the data better than the Poisson model, or vice-versa, not that zero-inflation is present or absent.
What Should We Do?

- Use Methods That Allow the zero inflation parameter to pass through zero:
  - Dietz and Böhning (1999)
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- Diagrammatic methods?
An idea in the very early stages of development
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Simply fit a (say) Poisson Model and a Zero-Inflated Poisson model and plot the densities of the observations against each other
The Trajan Data

- Response: Number of roots produced by an explant of the apple *Trajan* under micropropogation
- Covariates:
  - *Hormone*: Factor with four levels, not significant for neither mean nor zero-inflation parameter.
  - *Period*: Factor with two levels, significant for both mean and zero-inflation parameter.
But First: What happened if there is not zero inflation, but the period is significant for the mean, but hormone is not.
But First: What happened if there is not zero inflation, but the period is significant for the mean, but hormone is not.
Now we colour by hormone:
Back to the real response variable, recall there is zero-inflation, but only period is significant.
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The Trajan Data

Period Distinguished

Poisson Densities (with Jitter)
ZIP Densities (with Jitter)

Hormone Distinguished

Poisson Densities (with Jitter)
ZIP Densities (with Jitter)
The Mis-use of the Vuong Test as a Test of Zero-Inflation

The Trajan Data

Period Distinguished

Hormone Distinguished

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Thank You!!